Grade 7/8 Math Circles
October 2/3/4/5
Constructable Numbers Solutions

Exercise Solutions

1. Create the number four and then construct a triangle where one side has length 2 and another has length 4.

   Solution: Similar to our construction in example 2, draw a straight line. We have two ways to construct the number 4

   (i) Measure your unit length from before, and place it on the line. Mark each end of the compass. Shift your compass over so one end is on one of your marks, and the other is further down the line. Again, mark this new point on your line. Do this twice more to get 4.

   (ii) Since we've already created 2, and $4 = 2 + 2$, measure your number 2 from the example. Then copy this just once to get 4.

   Once you have 4, draw two lines that intersect. One point of your triangle will be the intersection. With your compass, measure your length 2. Place one end of the compass on the intersection and the other on one of the lines. Mark this point on the line. Do the same thing with your length 4 on the other line. Connect the two points that you created to form your triangle!

2. Draw the lengths $\frac{5}{3}$ and $\frac{4}{2}$. What do you notice about the length $\frac{4}{2}$.

   Solution: Since we need a length of 5, we'll need to construct it first. Once this is done, we follow the steps from example 3.

   (a) Draw the length of 3.

   (b) In any direction that isn't the same as the first line, draw a line of length 5 from the right end of the first line.

   (c) Complete the triangle.

   (d) On the line of length 3, measure out a unit length from the left corner.

   (e) Draw a line parallel to the line of length 5 that touches the end of the unit length. This line has length $\frac{5}{3}$.

We perform the same steps for the length $\frac{4}{2}$. If everything has been constructed correctly, we should be able to notice that $\frac{4}{2}$ is the same length as the length 2 that we previously constructed. We should expect this since $\frac{4}{2} = 2$!

3. Draw a right-angled triangle where two of the lengths are 3 and 4.

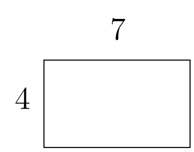
   Solution: Begin by following the steps in example 5 to draw the right angle. Once you have your perpendicular lines, on one line measure a length of 3 from the intersection. That is, copy your length of 3 using your compass, place one tip of your compass on the intersection of the two lines, and the other tip on one of the lines. On the other line, measure a length of 4 from the intersection. Connect the two lines to get a right-angled triangle.

4. Ethan and Khanh are building a shelf. They went to the hardware store and bought a piece of wood that is 4 inches wide and 7 inches long. They want to transform the wood so that it is only 2 inches wide. What will the length of their shelf be?

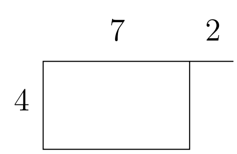
   If you can, verify that your answer is correct using algebra.

   Solution: We follow the steps from example 6.

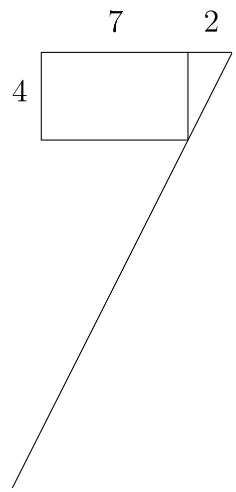
   (a) Construct a rectangle that has side lengths 4 and 7



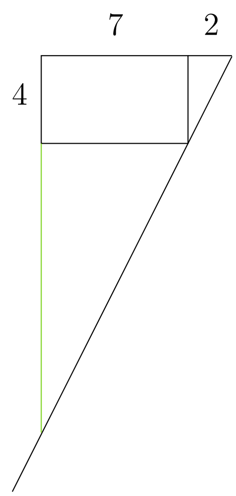
   (b) Extend the top of the rectangle by 2



   (c) Connect the corner of the rectangle with the end of the line, and extend the line as far as you can.

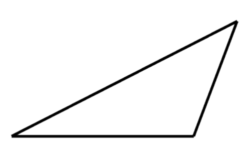


(d) Extend the far side of the rectangle straight down until you reach the diagonal. This extension is the missing length.

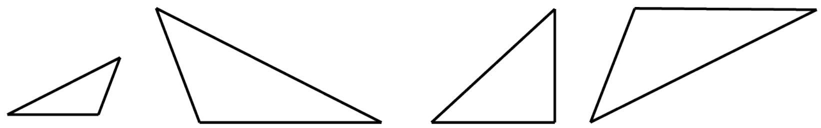


Using algebra, we want the areas of the rectangle to be the same. The area of the original rectangle is $4 \times 7 = 28$. The area of the second is $2 \times length$. So we have $28 = 2 \times length$ and so our length should be 14. We have many ways to check if our line has length 14. We can (a) measure the unit length 14 times, (b) measure our length 2 7 times, or (c) measure our length 7 twice.

5. Consider the following triangle:



Which of the following triangles are congruent to the above triangle?



Solution: The second and fourth triangles are congruent to the above triangle.

Problem Set Solutions

1. (a) Construct the numbers 3, 4, and 5.

   (b) Draw a triangle where two of the sides have length 3 and 5.

   Solution:

   (a) If you completed all the exercises, then all three of these lengths will already be constructed. However, if you haven't we can follow the steps from example 2 to construct each of these numbers.

   (b) Draw two lines that intersect. One point of your triangle will be the intersection. With your compass, measure your length 3. Place one end of the compass on the intersection and the other on one of the lines. Mark this point on the line. Do the same thing with your length 5 on the other line. Connect the two points that you created to form your triangle!

2. (a) Practice drawing parallel lines by drawing a random line and point, and then drawing a parallel line.

   (b) Re-do example 3, but instead of sliding your straightedge in step (v), draw a parallel line using the method that you just learned.

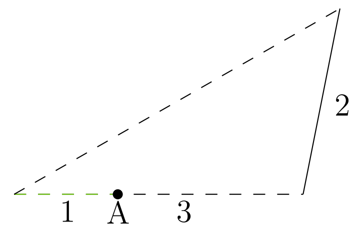
   Caution: It's very easy for your paper to become crowded when drawing parallel lines at the same time as constructing a fraction. Try to erase any lines that you no longer need.

   (c) Draw the fraction $\frac{5}{4}$ and use the above method to draw the parallel line in step (v) of constructing fractions.

   Solution:

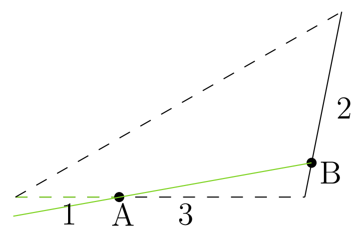
   (a) Follow the steps in the example to create your parallel lines.

   (b) We take our triangle from step (iv) in example 3

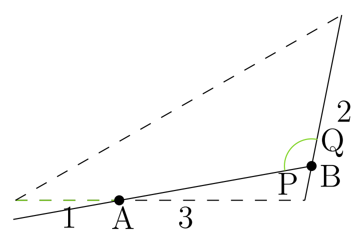


We want to draw a line parallel to the line of length 2 through the left side of the unit length, which is labelled A.

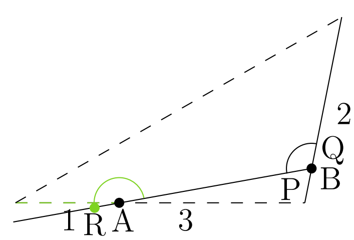
i. Pick a point on the line and label it 'B'. Draw a new line through the points A and B.



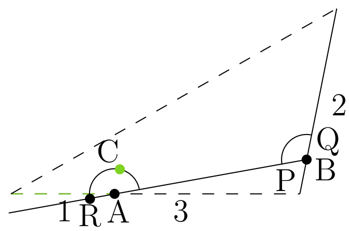
ii. Using your compass, place the tip of the compass point B, and the other end on the line you just drew. Draw an arc from the new line to the old line. Label each end of the arc 'P' and 'Q'.



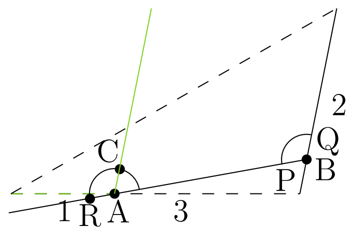
iii. Without adjusting your compass, plce the tip on A and draw another arc. Label the point where the arc and line meet R.



iv. Adjust your compass so one tip is on P and the other is on Q. Now, without adjusting the compass, place one tip on R. Place the other end so it is on the arc. Label this point on the second arc 'C'.



v. Draw a line through A and C.



We've now created our parallel line.

(c) We'll leave the construction of $\frac{5}{4}$ for the reader to try without the solution.

3. Construct the following fractions and use your constructions determine which pairs of fractions are equal.

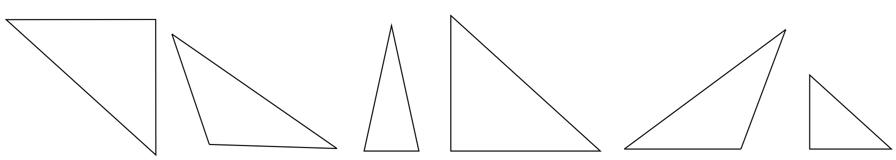
   (a) $\frac{5}{3}$ and $\frac{4}{2}$. (b) $\frac{1}{3}$ and $\frac{2}{6}$. (c) $\frac{1}{3}$ and $\frac{2}{3}$. (d) $\frac{3}{3}$ and 1.

   Solution: Since the steps to construct the fractions is the same every time, we omit the solutions for each of the constructions. You should, however, notice that the fractions in (b) and (d) are the same length.

4. Now that we can draw both perpendicular lines and parallel lines, draw a rectangle without using a protractor.

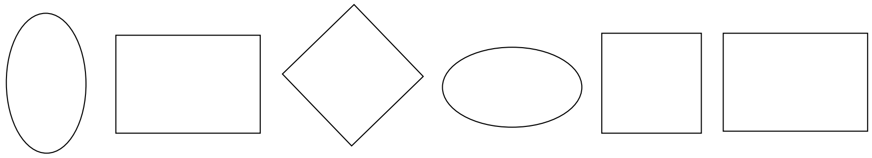
   Solution: Again, since the steps are exactly the same as in the examples, we'll omit the solution to the construction.

5. Find pairs of congruent triangles below:



   Solution: The first and fourth triangles are congruent, and the second and fifth triangles are congruent.

6. Which of the following shapes are congruent?

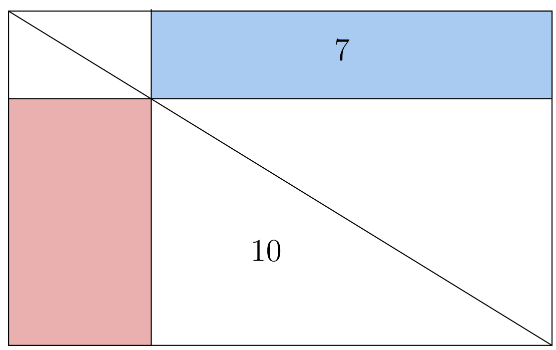


Solution: The first and fourth shapes are congruent and the second and last shapes are congruent.

7. Kateryna is making a dance floor out of wood. She has a rectangular piece of wood that has side lengths 4 and 6 and she wants her floor to have a width of 3. Using geometry, how long will her dance floor be if she doesn't waste any wood? Make sure you accurately construct your rectangles using either the method from question 4 or by using a protractor.

   Solution: Your initial rectangle should have side lengths of 4 and 6, and you should extend the top of the rectangle by a length of 3. Then follow the steps from example 6.

8. Consider the following diagram:



   The area of the blue rectangle is 7. The area of one big triangle is 10. The combined area of all the shapes is 40.

   (a) What is the area of the second big triangle?
   (b) What is the area of the pink rectangle? How do you know?
   (c) What is the area of one small triangle?

   Solution:

   (a) Since the two big triangles are congruent, the second big triangle also has an area of 10.
   (b) We saw in the justification of our work in example 6 that the two small rectangles formed should have the same area. Thus, the area of the pink rectangle is 7.

(c) Since the combined area of all the shapes is 40, each large triangle has an area of 10, and each rectangle has an area of 7, we have that the two small triangles together has an area of

$$40 - 2 \times 10 - 2 \times 7 = 6$$

To get each triangle, we divide the total area by 2, so

$$6 \div 2 = 3$$

And thus each small triangle has an area of 3.

Extension Solutions

1. (i) Identify which numbers are the legs, and which number is the hypotenuse in the triangle from the extension.

   (ii) Suppose we have a right-angled triangle with legs of length 5 and 12. Using the *Pythagorean Theorem*, what is c^2? What is c?

   Hint: To find out what c is, take the *square root* of c^2. The symbol for square root on your calculator is $\sqrt{\ }$

   (iii) Repeat part (i) using legs with length 8 and 15.

   Solution:

   (i) The legs are the sides with lengths 3 and 4. The hypotenuse is the side with length 5.

   (ii) By the Pythagorean theorem, we have

   $$c^2 = 5^2 + 12^2 = 169$$

   Then, by taking the square root, we have

   $$c = \sqrt{c^2} = \sqrt{169} = 13$$

   (iii) Using the Pythagorean theorem, we have $c^2 = 289$. Taking the square root, we have $c = 17$.

2. (a) We leave the right-angled triangle to be drawn by the reader, following the steps to draw perpendicular lines, and then to draw a triangle with 2 side lengths.

(b) If you construct a right-angled triangle with legs of lengths 4 and 2, then you construct the number $\sqrt{20}$. If you construct a right-angled triangle with legs of lengths 3 and 1, then you construct the number $\sqrt{10}$.

(c) Constructing fractions with irrational numbers is no different than constructing those with regular fractions as side lengths.

One should notice that the length of $\frac{\sqrt{2}}{2}$ is the same as $\frac{1}{\sqrt{2}}$.